

Elementary quotient completion

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Abstract

We extend the notion of exact completion on a weakly lex category to elementary doctrines. We show how any such doctrine admits an elementary quotient completion, which freely adds effective quotients and extensional equality. We note that the elementary quotient completion can be obtained as the composite of two free constructions: one adds effective quotients, and the other forces extensionality of maps. We also prove that each construction preserves comprehensions.

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1 Introduction

Constructions for completing a category by quotients has been widely studied in category theory. The main instance is the so-called exact completion in (Carboni and Magno 1982; Carboni and Vitale 1998) which shows how to add, in a finitary way, quotients that are defined as effective coequalizers of monic equivalence relations to suitable categories by turning them into exact categories.

The use of quotient completion is also pervasive in interactive theorem proving where proofs are performed in appropriate systems of formalized mathematics in a computer-assisted way. Indeed the use of a quotient completion is rather compulsory when mathematics is formalized within an intensional type theory, such as the Calculus of (Co)Inductive Constructions (Coquand 1990; Coquand and Paulin-Mohring 1990) or Martin-Löf's type theory (Nordström, Petersson, and Smith 1990). In such a context, the abstract construction of quotient completion provides a formal framework where to combine the usual practice of (extensional) mathematics, with the need of formalizing it in an intensional theory with strong decidable properties (such as decidable type-checking) on which to perform the extraction of algorithmic contents from proofs.

To make explicit the use of quotient completion in the formalization of constructive mathematics, in (Maietti 2009) it has been included as a part of the definition of constructive foundation. According to (Maietti 2009), a constructive foundation must be a

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two-level theory as first argued in (Maietti and Sambin 2005): it must be equipped with an intensional level, which can be represented by a suitable starting category \mathcal{C} , and an extensional level that can be seen as (a fragment of) the internal language of a suitable quotient completion of \mathcal{C} . As investigated in (Maietti and Rosolini 2012), some examples of quotient completion performed on intensional theories, such as the intensional level of the minimalist foundation in (Maietti 2009), or the Calculus of Constructions, do not fall under the known constructions of exact completion given that the corresponding type theoretic categories closed under quotients are not exact.

In (Maietti and Rosolini 2012) we studied the abstract categorical structure behind such quotient completions. To this purpose we introduced the notion of equivalence relation and quotient relative to a suitable fibered poset and produced a free construction adding effective quotients—hence the name elementary quotient completion—to elementary doctrines.

In the present paper we isolate the basic components of the free constructions in (Maietti and Rosolini 2012). After recalling the basic notions required in the sequel, we show how to add effective quotients freely to an elementary doctrine in the sense of (Lawvere 1970), a fibered infsemilattice on a cartesian category, endowed with equality. Separately, we describe how to force extensional equality of maps to (the base of) an elementary doctrine. Then we prove that the two constructions can be combined to give the elementary quotient completion. Finally we check that the exact completion of a weakly lex cartesian category is an instance of the elementary quotient completion while the regular completion of a weakly lex cartesian category is an instance of a rather different construction.

2 Doctrines

The notion of a doctrine is the basic categorical concept we adopt to analyse quotients. It was introduced, in a series of seminal papers, by F.W. Lawvere to synthesize the structural properties of logical systems, see (Lawvere 1969a; Lawvere 1969b; Lawvere 1970), see also (Lawvere and Rosebrugh 2003) for a unified survey. Lawvere’s crucial intuition was to consider logical languages and theories as fibrations to study their 2-categorical properties, *e.g.* connectives and quantifiers are determined by structural adjunctions. That approach proved extremely fruitful, see (Makkai and Reyes 1977; Lambek and Scott 1986; Jacobs 1999; Taylor 1999; van Oosten 2008) and the references therein.

Taking advantage of the algebraic presentation of logic by fibrations, we first introduce a general notion of elementary doctrine which we found appropriate to study the notion of quotient of an equivalence relation, see (Maietti and Rosolini 2012).

2.1 DEFINITION. An *elementary doctrine* is a functor $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ from (the opposite of) a category \mathcal{C} with binary products to the category of inf-semilattices and homomorphisms such that, for every object A in \mathcal{C} , there is an object δ_A in $P(A \times A)$ and

(i) the assignment

$$\mathcal{A}_{\langle \text{id}_A, \text{id}_A \rangle}(\alpha) := P_{\text{pr}_1}(\alpha) \wedge_{A \times A} \delta_A$$

for α in $P(A)$ determines a left adjoint to $P_{\langle \text{id}_A, \text{id}_A \rangle}: P(A \times A) \rightarrow P(A)$ —the action of a doctrine P on an arrow is written as P_f

(ii) for every map $e := \langle \text{pr}_1, \text{pr}_2, \text{pr}_2 \rangle: X \times A \rightarrow X \times A \times A$ in \mathcal{C} , the assignment

$$\mathcal{A}_e(\alpha) := P_{\langle \text{pr}_1, \text{pr}_2 \rangle}(\alpha) \wedge_{A \times A} P_{\langle \text{pr}_2, \text{pr}_3 \rangle}(\delta_A)$$

for α in $P(X \times A)$ determines a left adjoint to $P_e: P(X \times A \times A) \rightarrow P(X \times A)$.

2.2 REMARK. (a) In case \mathcal{C} has a terminal object, conditions (ii) entails condition (i).

(b) One has that $\top_A \leq P_{\langle \text{id}_A, \text{id}_A \rangle}(\delta_A)$ and $\delta_A \leq P_{f \times f}(\delta_B)$ for $f: A \rightarrow B$.

2.3 REMARK. For α_1 in $P(X_1 \times Y_1)$ and α_2 in $P(X_2 \times Y_2)$, it is useful to introduce a notation like $\alpha_1 \boxtimes \alpha_2$ for the object

$$P_{\langle \text{pr}_1, \text{pr}_3 \rangle}(\alpha_1) \wedge P_{\langle \text{pr}_2, \text{pr}_4 \rangle}(\alpha_2)$$

in $P(X_1 \times X_2 \times Y_1 \times Y_2)$ where $\text{pr}_i, i = 1, 2, 3, 4$, are the projections from $X_1 \times X_2 \times Y_1 \times Y_2$ to each of the four factors.

Condition 2.1(ii) is to request that $\delta_{A \times B} = \delta_A \boxtimes \delta_B$ for every pair of objects A and B in \mathcal{C} .

2.4 EXAMPLES. (a) The standard example of an indexed poset is the fibration of subobjects. Consider a category \mathcal{X} with products and pullbacks. The functor $S: \mathcal{X}^{\text{op}} \rightarrow \mathbf{InfSL}$ assigns to any object A in \mathcal{X} the poset $S(A)$ of subobjects of A in \mathcal{X} . For an arrow $f: B \rightarrow A$, the assignment that maps a subobject in $S(A)$ to that represented by the left-hand arrow in any pullback along f of its produces a functor $S_f: S(A) \rightarrow S(B)$ that preserves products.

The elementary structure is provided by the diagonal maps.

(b) The leading logical example is the indexed order $LT: \mathcal{V}^{\text{op}} \rightarrow \mathbf{InfSL}$ given by the Lindenbaum-Tarski algebras of well-formed formulae of a theory \mathcal{T} with equality in a first order language \mathcal{L} .

The domain category is the category \mathcal{V} of lists of variables and term substitutions:

object of \mathcal{V} are lists of distinct variables $\vec{x} = (x_1, \dots, x_n)$

arrows are lists of substitutions¹ for variables $[\vec{t}/\vec{y}]: \vec{x} \rightarrow \vec{y}$ where each term t_j in \vec{t} is built in \mathcal{L} on the variables x_1, \dots, x_n

composition $\vec{x} \xrightarrow{[\vec{t}/\vec{y}]} \vec{y} \xrightarrow{[\vec{s}/\vec{z}]} \vec{z}$ is given by simultaneous substitutions

$$\vec{x} \xrightarrow{[s_1[\vec{t}/\vec{y}]/z_1, \dots, s_k[\vec{t}/\vec{y}]/z_k]} \vec{z}$$

¹We shall employ a vector notation for lists of terms in the language as well as for simultaneous substitutions such as $[\vec{t}/\vec{y}]$ in place of $[t_1/y_1, \dots, t_m/y_m]$.

The product of two objects \vec{x} and \vec{y} is given by a(ny) list \vec{w} of as many distinct variables as the sum of the number of variables in \vec{x} and of that in \vec{y} . Projections are given by substitution of the variables in \vec{x} with the first in \vec{w} and of the variables in \vec{y} with the last in \vec{w} .

The functor $LT: \mathcal{V}^{\text{op}} \longrightarrow \mathbf{InfSL}$ is given as follows: for a list of distinct variables \vec{x} , the category $LT(\vec{x})$ has

objects equivalence classes of well-formed formulae of \mathcal{L} with no more free variables than x_1, \dots, x_n with respect to provable reciprocal consequence $W \dashv\vdash_{\mathcal{T}} W'$ in \mathcal{T} .

arrows $[W] \rightarrow [V]$ are the provable consequences $W \vdash_{\mathcal{T}} V$ in \mathcal{T} for some pair of representatives (hence for any pair)

composition is given by the cut rule in the logical calculus

identities $[W] \rightarrow [W]$ are given by the logical rules $W \vdash_{\mathcal{T}} W$

For a list of distinct variables \vec{x} , the category $LT(\vec{x})$ has finite limits: a terminal object is $\vec{x} = \vec{x}$ and products are given by conjunctions of formulae.

(c) Consider a cartesian category \mathcal{S} with weak pullbacks. Another example of elementary doctrine which appears *prima facie* very similar to previous example (a) is given by the functor of *weak subobjects* $\Psi: \mathcal{S}^{\text{op}} \longrightarrow \mathbf{InfSL}$ which evaluates as the poset reflection of each comma category \mathcal{S}/A at each object A of \mathcal{S} , introduced in (Grandis 2000).

The apparently minor difference between the present example and example (a) depends though on the possibility of factoring an arbitrary arrow as a retraction followed by a monomorphism: for instance this can be achieved in the category **Set** of sets and functions thanks to the Axiom of Choice, see *loc.cit.*

It is possible to express precisely how the examples are related once we consider the 2-category **ED** of elementary doctrines:

the 1-arrows are pairs (F, b)

$$\begin{array}{ccc} \mathcal{C}^{\text{op}} & & \\ \downarrow F & \searrow P & \\ & & \mathbf{InfSL} \\ \mathcal{D}^{\text{op}} & \nearrow R & \\ & \downarrow b & \end{array}$$

where the functor F preserves products and, for every object A in \mathcal{C} , the functor $b_A: P(A) \rightarrow R(F(A))$ preserves all the structure. More explicitly, b_A preserves finite meets and, for every object A in \mathcal{C} ,

$$b_{A \times A}(\delta_A) = R_{(F(\text{pr}_1), F(\text{pr}_2))}(\delta_{F(A)}). \quad (1)$$

the 2-arrows are natural transformations θ such that

$$\begin{array}{ccc} \mathcal{C}^{\text{op}} & & \\ \downarrow F & \searrow P & \\ & & \mathbf{InfSL} \\ \mathcal{D}^{\text{op}} & \nearrow R & \\ & \downarrow b & \end{array} \quad \begin{array}{c} \xrightarrow{\theta} \\ \xrightarrow{\leq} \end{array}$$

so that, for every object A in \mathcal{C} and every α in $P(A)$, one has $R_{\theta_A}(b_A(\alpha)) \leq c_A(\alpha)$.

2.5 EXAMPLES. (a) Given a theory \mathcal{T} with equality in a first order language, a 1-arrow $(F, b): LT \rightarrow S$ from the elementary doctrine $LT: \mathcal{V}^{\text{op}} \rightarrow \mathbf{InfSL}$ as in 2.4(a) into the elementary doctrine $S: \mathbf{Set}^{\text{op}} \rightarrow \mathbf{InfSL}$ as in 2.4(b) determines a model \mathfrak{M} of \mathcal{T} where the set underlying the interpretation is $F(x = x)$. In fact, there is an equivalence between the category $\mathbf{ED}(LT, S)$ and the category of models of \mathcal{T} and homomorphisms.

(b) Given a category \mathcal{X} with products and pullbacks, one can consider the two indexed posets: that of subobjects $S: \mathcal{X}^{\text{op}} \rightarrow \mathbf{InfSL}$, and the other $\Psi: \mathcal{X}^{\text{op}} \rightarrow \mathbf{InfSL}$, obtained by the poset reflection of each comma category \mathcal{X}/A , for A in \mathcal{X} . The inclusions of the poset $S(A)$ of subobjects over A into the poset reflection of \mathcal{X}/A extend to a 1-arrow from S to Ψ which is an equivalence exactly when every arrow in \mathcal{X} can be factored as a retraction followed by a monic.

3 Quotients in an elementary doctrine

The structure of elementary doctrine is suitable to describe the notions of an equivalence relation and of a quotient for such a relation.

3.1 DEFINITION. Given an elementary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$, an object A in \mathcal{C} and an object ρ in $P(A \times A)$, we say that ρ is a ***P -equivalence relation on A*** if it satisfies

reflexivity: $\delta_A \leq \rho$

symmetry: $\rho \leq P_{\langle \text{pr}_2, \text{pr}_1 \rangle}(\rho)$, for $\text{pr}_1, \text{pr}_2: A \times A \rightarrow A$ the first and second projection, respectively

transitivity: $P_{\langle \text{pr}_1, \text{pr}_2 \rangle}(\rho) \wedge P_{\langle \text{pr}_2, \text{pr}_3 \rangle}(\rho) \leq P_{\langle \text{pr}_1, \text{pr}_3 \rangle}(\rho)$, for $\text{pr}_1, \text{pr}_2, \text{pr}_3: A \times A \times A \rightarrow A$ the projections to the first, second and third factor, respectively.

In elementary doctrines as those presented in 2.4, P -equivalence relations coincide with the usual notion for those of the form (a) or (b); more interestingly, in cases like (c) a Ψ -equivalence relation is a pseudo-equivalence relation in \mathcal{S} in the sense of (Carboni and Magno 1982).

For $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ an elementary doctrine, the object δ_A is a P -equivalence relation on A . And for an arrow $f: A \rightarrow B$ in \mathcal{C} , the functor $P_{f \times f}: P(B \times B) \rightarrow P(A \times A)$ takes a P -equivalence relation σ on B to a P -equivalence relation on A . Hence, the ***P -kernel of $f: A \rightarrow B$*** , the object $P_{f \times f}(\delta_B)$ of $P_{A \times A}$ is a P -equivalence relation on A . In such a case, one speaks of $P_{f \times f}(\delta_B)$ as an ***effective P -equivalence relation***.

3.2 REMARK. A 1-arrow $(F, b): P \rightarrow R$ in \mathbf{ED} takes a P -equivalence relation on A to an R -equivalence relation on FA .

3.3 DEFINITION. Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ be an elementary doctrine. Let ρ be a P -equivalence relation on A . A ***quotient of ρ*** is a arrow $q: A \rightarrow C$ in \mathcal{C} such that $\rho \leq P_{q \times q}(\delta_C)$ and, for every arrow $g: A \rightarrow Z$ such that $\rho \leq P_{g \times g}(\delta_Z)$, there is a unique arrow $h: C \rightarrow Z$ such that $g = h \circ q$.

We say that such a quotient is **stable** if, in every pullback

$$\begin{array}{ccc} A' & \xrightarrow{q'} & C' \\ f' \downarrow & & \downarrow f \\ A & \xrightarrow{q} & C \end{array}$$

in \mathcal{C} , the arrow $q': A' \rightarrow C'$ is a quotient.

3.4 REMARK. Note that the inequality $\rho \leq P_{q \times q}(\delta_C)$ in 3.3 becomes an identity exactly when ρ is effective.

In the elementary doctrine $S: \mathcal{X}^{\text{op}} \rightarrow \mathbf{InfSL}$ obtained from a category \mathcal{X} with products and pullbacks as in 2.4(a), a quotient of the S -equivalence relation $[r: R \twoheadrightarrow A \times A]$ is precisely a coequalizer of the pair of

$$R \begin{array}{c} \xrightarrow{\text{pr}_1 \circ r} \\ \xrightarrow{\text{pr}_2 \circ r} \end{array} A$$

In particular, all S -equivalence relations have stable, effective quotients if and only if the category \mathcal{C} is exact.

Similarly, in the elementary doctrine $\Psi: \mathcal{S}^{\text{op}} \rightarrow \mathbf{InfSL}$ obtained from a cartesian category \mathcal{X} with weak pullbacks as in 2.4(c), a quotient of the Ψ -equivalence relation $[r: R \rightarrow A \times A]$ is precisely a coequalizer of the pair of

$$R \begin{array}{c} \xrightarrow{\text{pr}_1 \circ r} \\ \xrightarrow{\text{pr}_2 \circ r} \end{array} A$$

In particular, all Ψ -equivalence relations have quotients which are stable if and only if the category \mathcal{C} is exact.

3.5 DEFINITION. Given an elementary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ and a P -equivalence relation ρ on an object A in \mathcal{C} , the poset of descent data \mathcal{Des}_ρ is the sub-poset of $P(A)$ on those α such that

$$P_{\text{pr}_1}(\alpha) \wedge_{A \times A} \rho \leq P_{\text{pr}_2}(\alpha),$$

where $\text{pr}_1, \text{pr}_2: A \times A \rightarrow A$ are the projections.

3.6 REMARK. Given an elementary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$, for $f: A \rightarrow B$ in \mathcal{C} , let χ be the P -kernel $P_{f \times f}(\delta_B)$. The functor $P_f: P(B) \rightarrow P(A)$ applies $P(B)$ into \mathcal{Des}_χ .

3.7 DEFINITION. Given an elementary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ and an arrow $f: A \rightarrow B$ in \mathcal{C} , let χ be the P -kernel $P_{f \times f}(\delta_B)$. The arrow f is **descent** if the (obviously faithful) functor $P_f: P(B) \rightarrow \mathcal{Des}_\chi$ is also full. The arrow f is **effective descent** if the functor $P_f: P(B) \rightarrow \mathcal{Des}_\chi$ is an equivalence.

Consider the 2-full 2-subcategory **QED** of **ED** whose objects are elementary doctrines $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ with descent quotients of P -equivalence relations.

The **1-arrows** are those pairs (F, b) in **ED**

$$\begin{array}{ccc}
 \mathcal{C}^{\text{op}} & & \\
 \downarrow F & \searrow P & \\
 \mathcal{D}^{\text{op}} & \xrightarrow{R} & \mathbf{InfSL}
 \end{array}
 \quad
 \begin{array}{ccc}
 & \downarrow b & \\
 & \nearrow R & \\
 & &
 \end{array}$$

such that F preserves quotients in the sense that, if $q: A \rightarrow C$ is a quotient of a P -equivalence relation ρ on A , then $Fq: FA \rightarrow FC$ is a quotient of the R -equivalence relation $R_{\langle F(\text{pr}_1), F(\text{pr}_2) \rangle}(b_{A \times A}(\rho))$ on FA .

4 Completing with quotients as a free construction

It is a simple construction that produces an elementary doctrine with quotients. We shall present it in the following and prove that it satisfies a universal property.

Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ be an elementary doctrine for the rest of the section. Consider the category \mathcal{R}_P of “equivalence relations of P ”:

an object of \mathcal{R}_P is a pair (A, ρ) such that ρ is a P -equivalence relation on A

an arrow $f: (A, \rho) \rightarrow (B, \sigma)$ is an arrow $f: A \rightarrow B$ in \mathcal{C} such that $\rho \leq_{A \times A} P_{f \times f}(\sigma)$ in $P(A \times A)$.

Composition is given by that of \mathcal{C} , and identities are the identities of \mathcal{C} .

The indexed poset $(P)_q: \mathcal{R}_P^{\text{op}} \rightarrow \mathbf{InfSL}$ on \mathcal{R}_P will be given by categories of descent data: on an object (A, ρ) it is defined as

$$(P)_q(A, \rho) := \mathcal{Des}_\rho$$

and the following lemma is instrumental to give the assignment on arrows using the action of P on arrows.

4.1 LEMMA. *With the notation used above, let (A, ρ) and (B, σ) be objects in \mathcal{R}_P , and let β be an object in \mathcal{Des}_σ . If $f: (A, \rho) \rightarrow (B, \sigma)$ is an arrow in \mathcal{R}_P , then $P_f(\beta)$ is in \mathcal{Des}_ρ .*

Proof. Since β is in \mathcal{Des}_σ , it is

$$P_{\text{pr}'_1}(\beta) \wedge \sigma \leq_{B \times B} P_{\text{pr}'_2}(\beta)$$

where $\text{pr}'_1, \text{pr}'_2: B \times B \rightarrow B$ are the two projections. Hence

$$P_{f \times f}(P_{\text{pr}'_1}(\beta)) \wedge P_{f \times f}(\sigma) \leq_{A \times A} P_{f \times f}(P_{\text{pr}'_2}(\beta))$$

as $P_{f \times f}$ preserves the structure. Since $\rho \leq_{A \times A} P_{f \times f}(\sigma)$,

$$P_{\text{pr}_1}(P_f(\beta)) \wedge \rho \leq_{A \times A} P_{\text{pr}_2}(P_f(\beta))$$

where $\text{pr}_1, \text{pr}_2: A \times A \rightarrow A$ are the two projections. □

4.2 LEMMA. *With the notation used above, $(P)_q: \mathcal{R}_P^{op} \rightarrow \mathbf{InfSL}$ is an elementary doctrine.*

Proof. For (A, ρ) and (B, σ) in \mathcal{R}_P let $\text{pr}_1, \text{pr}_3: A \times B \times A \times B \rightarrow A$ and $\text{pr}_2, \text{pr}_4: A \times B \times A \times B \rightarrow B$ be the four projections. As a meet of two P -equivalence relations on $A \times B$, the P -equivalence relation

$$\rho \boxtimes \sigma := P_{\langle \text{pr}_1, \text{pr}_3 \rangle}(\rho) \wedge_{A \times B \times A \times B} P_{\langle \text{pr}_2, \text{pr}_4 \rangle}(\sigma)$$

provides an object $(A \times B, \rho \boxtimes \sigma)$ in \mathcal{R}_P which, together with the arrows determined by the two projections from $A \times B$, is a product of (A, ρ) and (B, σ) in \mathcal{R}_P .

For each (A, ρ) , the sub-poset $\mathcal{Des}_\rho \subseteq P(A)$ is closed under finite meets.

For an object (A, ρ) in \mathcal{R}_P , consider the object $P_{\langle \text{pr}_1, \text{pr}_2 \rangle}(\rho)$ in $P(A \times A \times A \times A)$. It is easy to see that it is in $\mathcal{Des}_{\rho \boxtimes \rho}$. Such objects satisfy 2.1 (i) and (ii): the assignment $((\mathcal{I})_q)_{\Delta_A}(\alpha) := P_{\text{pr}_1}(\alpha) \wedge_{A \times A} \rho$, for α in \mathcal{Des}_ρ , gives the left adjoint $((\mathcal{I})_q)_{\Delta_A}$ for $((P)_q)_{\Delta_A}$. Indeed, let θ be in $\mathcal{Des}_{\rho \boxtimes \rho}$ such that $\alpha \leq_{(A, \rho)} ((P)_q)_{\Delta_A}(\theta)$, i.e. $\alpha \leq_A P_{\Delta_A}(\theta)$. Thus $\mathcal{I}_{\Delta_A}(\alpha) \leq_A \theta$ and one has

$$\begin{aligned} P_{\text{pr}_1}(\alpha) \wedge P_{\langle \text{pr}_1, \text{pr}_2 \rangle}(\delta_A) \wedge P_{\langle \text{pr}_2, \text{pr}_3 \rangle}(\rho) &\leq_{A \times A \times A} P_{\langle \text{pr}_1, \text{pr}_2 \rangle}(\theta) \wedge P_{\langle \text{pr}_2, \text{pr}_3 \rangle}(\rho) \\ &\leq_{A \times A \times A} P_{\langle \text{pr}_1, \text{pr}_3 \rangle}(\theta) \end{aligned}$$

for $\text{pr}_i: A \times A \times A \rightarrow A$, $i = 1, 2, 3$, the projections. Hence $P_{\text{pr}_1}(\alpha) \wedge \rho \leq_{A \times A} \theta$, i.e. $((\mathcal{I})_q)_{\Delta_A}(\alpha) \leq_{(A \times A, \rho \boxtimes \rho)} \theta$. It is easy to prove the converse that, if $((\mathcal{I})_q)_{\Delta_A}(\alpha) \leq \theta$, then $\alpha \leq ((P)_q)_{\Delta_A}(\theta)$. The proof of condition 2.1(ii) is similar. \square

There is an obvious 1-arrow $(J, j): P \rightarrow (P)_q$ in **ED**, where $J: \mathcal{C}^{op} \rightarrow \mathcal{R}_P$ sends an object A in \mathcal{C} to (A, δ_A) and an arrow $f: A \rightarrow B$ to $f: (A, \delta_A) \rightarrow (B, \delta_B)$ since $\delta_A \leq_{A \times A} P_{f \times f}(\delta_B)$, and $j_A: P(A) \rightarrow (P)_q(A, \delta_A)$ is the identity since, by definition,

$$(P)_q(A, \delta_A) = \mathcal{Des}_{\delta_A} = P(A).$$

It is immediate to see that J is full and faithful and that (J, j) is a change of base.

4.3 REMARK. Note that an object of the form (A, δ_A) in \mathcal{R}_P is projective with respect to quotients of $(P)_q$ -equivalence relation, and that every object in \mathcal{R}_P is a quotient of a $(P)_q$ -equivalence relation on such a projective.

4.4 LEMMA. *With the notation used above, $(P)_q: \mathcal{R}_P^{op} \rightarrow \mathbf{InfSL}$ has descent quotients of $(P)_q$ -equivalence relations. Moreover, quotients are stable and effective descent, and P -equivalence relations are effective.*

Proof. Since the sub-poset $\mathcal{Des}_\rho \subseteq P(A)$ is closed under finite meets, a $(P)_q$ -equivalence relation τ on (A, ρ) is also a P -equivalence relation on A . It is easy to see that $\text{id}_A: (A, \rho) \rightarrow (A, \tau)$ is a descent quotient since $\rho \leq_{A \times A} \tau$ —actually, effectively so. It follows immediately that τ is the P -kernel of the quotient $\text{id}_A: (A, \rho) \rightarrow (A, \tau)$. To see that it is also stable, suppose

$$\begin{array}{ccc} (B, v) & \xrightarrow{f'} & (A, \rho) \\ \downarrow g & & \downarrow \text{id}_A \\ (C, \sigma) & \xrightarrow{f} & (A, \tau) \end{array}$$

is a pullback in \mathcal{R}_P . So in the commutative diagram

$$\begin{array}{ccccc}
 & & f & & \\
 (C, \delta_C) & \xrightarrow{\quad} & (B, v) & \xrightarrow{f'} & (A, \rho) \\
 & \searrow \text{id}_C & \downarrow g & & \downarrow \text{id}_A \\
 & & (C, \sigma) & \xrightarrow{f} & (A, \tau)
 \end{array}$$

there is a fill-in map $h: (C, \delta_C) \rightarrow (B, v)$. It is now easy to see that $g: (B, v) \rightarrow (C, \sigma)$ is a quotient. \square

We can now prove that there is a left bi-adjoint to the forgetful 2-functor $U: \mathbf{QED} \rightarrow \mathbf{ED}$.

4.5 THEOREM. *For every elementary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$, pre-composition with the 1-arrow*

$$\begin{array}{ccc}
 \mathcal{C}^{\text{op}} & \xrightarrow{P} & \mathbf{InfSL} \\
 J \downarrow & j \downarrow & \uparrow (P)_q \\
 \mathcal{R}_P^{\text{op}} & \xrightarrow{(P)_q} & \mathbf{InfSL}
 \end{array}$$

in \mathbf{ED} induces an essential equivalence of categories

$$- \circ (J, j): \mathbf{QED}((P)_q, Z) \equiv \mathbf{ED}(P, Z) \quad (2)$$

for every Z in \mathbf{QED} .

Proof. Suppose Z is a doctrine in \mathbf{QED} . As to full faithfulness of the functor in (2), consider two pairs (F, b) and (G, c) of 1-arrows from $(P)_q$ to Z . By 4.3, the natural transformation $\theta: F \rightarrow G$ in a 2-arrow from (F, b) to (G, c) in \mathbf{QED} is completely determined by its action on objects in the image of J and $(P)_q$ -equivalence relations on these. And, since a quotient $q: U \rightarrow V$ of an Z -equivalence relation r on U is descent, $Z(V)$ is a full sub-poset of $Z(U)$. Thus essential surjectivity of the functor in (2) follows from 4.3. \square

Recall that, for an elementary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$, and for an object α in some $P(A)$, a **comprehensions** of α is a map $\{\alpha\}: X \rightarrow A$ in \mathcal{C} such that $P_{\{\alpha\}}(\alpha) = \top_X$ and, for every $f: Z \rightarrow A$ such that $P_f(\alpha) = \top_Z$ there is a unique map $g: Z \rightarrow X$ such that $f = \{\alpha\} \circ g$. One says that P **has comprehensions** if every α has a comprehension, and that P **has full comprehensions** if, moreover, $\alpha \leq \beta$ in $P(A)$ whenever $\{\alpha\}$ factors through $\{\beta\}$.

4.6 LEMMA. *Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ be an elementary doctrine. If P has comprehensions, then $(P)_q$ has comprehensions. Moreover, given a comprehension $\{\alpha\}: X \rightarrow A$ of α in $P(A)$, the map $J(\{\alpha\}): JX \rightarrow JA$ is a comprehension of $j_A(\alpha)$ if and only if $\delta_X = P_{\{\alpha\} \times \{\alpha\}}(\delta_A)$.*

Proof. Suppose (A, ρ) is in \mathcal{R}_P and α in $(P)_q(A, \rho) = \mathcal{Des}_\rho \subseteq P(A)$. Let $\{\alpha\}: X \rightarrow A$ be a comprehension in \mathcal{C} of α as an object of $P(A)$ and consider the object $(X, P_{\{\alpha\} \times \{\alpha\}}(\rho))$ in \mathcal{R}_P . Clearly $\{\alpha\}: (X, P_{\{\alpha\} \times \{\alpha\}}(\rho)) \rightarrow (A, \rho)$: we intend to show that that map is a comprehension of α as an object in $(P)_q(A, \rho)$. The following is a trivial computation in $\mathcal{Des}_{P_{\{\alpha\} \times \{\alpha\}}(\rho)} \subseteq P(X)$:

$$\top_X \leq P_{\{\alpha\}}(\alpha) = (P)_{q\{\alpha\}}(\alpha).$$

Suppose now that $f: (Z, \sigma) \rightarrow (A, \rho)$ is such that $tt_Z \leq (P)_{qf}(\alpha)$. Since $\{\alpha\}$ is a comprehension in \mathcal{C} , there is a unique map $g: Z \rightarrow X$ such that $f = \{\alpha\} \circ g$. To conclude, it is enough to show that $g: (Z, \sigma) \rightarrow (X, P_{\{\alpha\} \times \{\alpha\}}(\rho))$, but

$$\sigma \leq P_{f \times f}(\rho) = P_{g \times g}(P_{\{\alpha\} \times \{\alpha\}}(\rho)).$$

As for the second part of the statement, let α be in $P(A)$ and let $\{\alpha\}: X \rightarrow A$ be a comprehension of α in \mathcal{C} . Suppose, first, that $\delta_X = P_{\{\alpha\} \times \{\alpha\}}(\delta_A)$, and consider a map $f: (Z, \sigma) \rightarrow (A, \delta_A)$ such that $((P)_q)_f(\alpha) = \top_Z$. By definition of $(P)_q$, there is a unique map $g: Z \rightarrow X$ such $f = \{\alpha\} \circ g$ in \mathcal{C} . Thus

$$\sigma \leq P_{f \times f}(\delta_A) = P_{g \times g}P_{\{\alpha\} \times \{\alpha\}}(\delta_A) = P_{g \times g}(\delta_X).$$

Conversely, suppose $\{\alpha\}: (X, \delta_X) \rightarrow (A, \delta_A)$ in \mathcal{R}_P is a comprehension of α in $(P)_q$. Consider $\{\alpha\}: (X, P_{\{\alpha\} \times \{\alpha\}}(\delta_A)) \rightarrow (A, \delta_A)$. Since $((P)_q)_{\{\alpha\}}(\alpha) = P_{\{\alpha\}}(\alpha) = \top_X$, the map must factor through $\{\alpha\}: (X, \delta_X) \rightarrow (A, \delta_A)$, necessarily with the identity map. Hence the conclusion follows. \square

4.7 REMARK. When P has full comprehensions, the condition $\delta_X = P_{\{\alpha\} \times \{\alpha\}}(\delta_A)$ is ensured for all A and α .

Recall that the fibration of vertical maps on the category of points freely adds comprehensions to a given fibration producing an indexed poset in case the given fibration is such, see (Jacobs 1999). In our case of interest, for a doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$, the indexed poset consists of the base category \mathcal{G}_P where

an object is a pair (A, α) where A is in \mathcal{C} and α is in $P(A)$

an arrow $f: (A, \alpha) \rightarrow (B, \beta)$ is an arrow $f: A \rightarrow B$ in \mathcal{C} such that $\alpha \leq P_f(\beta)$.

The category \mathcal{G}_P has products and there is a natural embedding $I: \mathcal{C} \rightarrow \mathcal{G}_P$ which maps A to (A, \top_A) . The indexed functor extends to $(P)_c: \mathcal{G}_P^{\text{op}} \rightarrow \mathbf{InfSL}$ along I by setting $(P)_c(A, \alpha) := \{\gamma \in P(A) \mid \gamma \leq \alpha\}$. Moreover, the comprehensions in $(P)_c$ are full. As an immediate corollary, we have the following.

4.8 THEOREM. *There is a left bi-adjoint to the forgetful 2-functor from the full 2-category of **QED** on elementary doctrines with comprehensions and descent quotients into the 2-category **ED** of elementary doctrines.*

Proof. The left bi-adjoint sends an elementary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ to the elementary doctrine $((P)_c)_q: \mathcal{R}_{(P)_c}^{\text{op}} \rightarrow \mathbf{InfSL}$. \square

5 Extensional equality

In (Maietti and Rosolini 2012), “extensional” models of constructive theories, presented as doctrines $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$, were obtained by forcing the equality of arrows $f, g: A \rightarrow B$ in the base category \mathcal{C} to correspond to the “provable” equality $\top_A \leq_A P_{\langle f, g \rangle}(\delta_B)$ in the fibre $P(A)$. We recall from (Jacobs 1999) the basic property that supports the notion of very strong equality for the case of an elementary doctrine.

5.1 PROPOSITION. *Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ be an elementary doctrine and let A be an object in \mathcal{C} . The diagonal $\langle \text{id}_A, \text{id}_A \rangle: A \rightarrow A \times A$ is a comprehension if and only if it is the comprehension of δ_A .*

5.2 DEFINITION. Given an elementary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ we say that it has **comprehensive diagonals** if every diagonal map $\langle \text{id}_A, \text{id}_A \rangle: A \rightarrow A \times A$ is a comprehension.

5.3 REMARK. In case \mathcal{C} has equalizers, one finds that P has comprehensive diagonals in the sense of (Maietti and Rosolini 2012).

Let $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ be an elementary doctrine for the rest of the section. Consider the category \mathcal{X}_P , the “extensional collapse” of P :

the objects of \mathcal{X}_P are the objects of \mathcal{C}

an arrow $[f]: A \rightarrow B$ is an equivalence class of arrows $f: A \rightarrow B$ in \mathcal{C} such that $\delta_A \leq_{A \times A} P_{f \times f}(\delta_B)$ in $P(A \times A)$ with respect to the equivalence which relates f and f' when $\delta_A \leq_{A \times A} P_{f \times f'}(\delta_B)$.

Composition is given by that of \mathcal{C} on representatives, and identities are represented by identities of \mathcal{C} .

The indexed inf-semilattice $(P)_x: \mathcal{X}_P^{\text{op}} \rightarrow \mathbf{InfSL}$ on \mathcal{X}_P will be given essentially by P itself; the following lemma is instrumental to give the assignment on arrows using the action of P on arrows.

5.4 LEMMA. *With the notation used above, let $f, g: A \rightarrow B$ be arrows in \mathcal{C} and β an object in $P(B)$. If $\delta_A \leq_{A \times A} P_{f \times g}(\delta_B)$, then $P_f(\beta) = P_g(\beta)$.*

Proof. Since P is elementary,

$$P_{\text{pr}'_1}(\beta) \wedge \delta_B \leq_{B \times B} P_{\text{pr}'_2}(\beta)$$

where $\text{pr}'_1, \text{pr}'_2: B \times B \rightarrow B$ are the two projections. Hence

$$P_{f \times g}(P_{\text{pr}'_1}(\beta)) \wedge P_{f \times g}(\sigma) \leq_{A \times A} P_{f \times g}(P_{\text{pr}'_2}(\beta))$$

and, by the hypothesis that $\delta_A \leq_{A \times A} P_{f \times g}(\delta_B)$,

$$P_{f \circ \text{pr}_1}(\beta) \wedge \delta_A \leq_{A \times A} P_{g \circ \text{pr}_2}(\beta)$$

where $\text{pr}_1, \text{pr}_2: A \times A \rightarrow A$ are the two projections. Taking P_{Δ_A} of both sides,

$$P_f(\beta) = P_f(\beta) \wedge \top_A = P_{\Delta_A}(P_{f \circ \text{pr}_1}(\beta)) \wedge P_{\Delta_A}(\delta_A) \leq P_{\Delta_A}(P_{g \circ \text{pr}_2}(\beta)) = P_g(\beta).$$

The other direction follows by symmetry. □

In other words, the elementary doctrine $P: \mathcal{C}^{op} \rightarrow \mathbf{InfSL}$ factors through the quotient functor $K: \mathcal{C}^{op} \rightarrow \mathcal{X}_P$. That induces a 1-arrow of **ED** from $(K, k): P \rightarrow (P)_x$ in **ED**, where k_A is the identity for A in \mathcal{C} .

Consider the full 2-subcategory **CED** of **ED** whose objects are elementary doctrines $P: \mathcal{C}^{op} \rightarrow \mathbf{InfSL}$ with comprehensive diagonals.

The following result is now obvious.

5.5 LEMMA. *With the notation used above, $(P)_x: \mathcal{X}_P^{op} \rightarrow \mathbf{InfSL}$ is an elementary doctrine with comprehensive diagonals.*

Also the following is easy.

5.6 THEOREM. *For every elementary doctrine $P: \mathcal{C}^{op} \rightarrow \mathbf{InfSL}$, pre-composition with the 1-arrow*

$$\begin{array}{ccc} \mathcal{C}^{op} & & \\ \downarrow K & \searrow P & \\ \mathcal{X}_P^{op} & \xrightarrow{(P)_x} & \mathbf{InfSL} \end{array}$$

$k \cdot$

in **ED** induces an essential equivalence of categories

$$- \circ (K, k): \mathbf{CED}((P)_x, Z) \equiv \mathbf{ED}(P, Z) \quad (3)$$

for every Z in **CED**.

We can now mention the explicit connection between the two free constructions we have considered. For that it is useful to prove the following two lemmata.

5.7 LEMMA. *Let $P: \mathcal{C}^{op} \rightarrow \mathbf{InfSL}$ be an elementary doctrine. The arrow $(K, k): P \rightarrow (P)_x$ preserves quotients, in the sense that if $q: A \rightarrow C$ is a quotient of the P -equivalence relation ρ in $P(A \times A)$, then $K(q): KA \rightarrow KC$ is a quotient of $K_{\langle K(\text{pr}_1), K(\text{pr}_2) \rangle}(k_{A \times A}(\rho))$. Therefore, if P has descent quotients of P -equivalence relations, then $(P)_x$ has descent quotients of $(P)_x$ -equivalence relations.*

Proof. Since K is a quotient functor, it preserves quotients of P -equivalence relations. Since the k -components of $(K, k): P \rightarrow (P)_x$ are identity functions, a $(P)_x$ -equivalence relation τ on A is also a P -equivalence relation on A . \square

5.8 LEMMA. *Let $P: \mathcal{C}^{op} \rightarrow \mathbf{InfSL}$ be an elementary doctrine. If P has comprehensions, then $(P)_x$ has comprehensions. Moreover $(K, k): P \rightarrow (P)_x$ preserves comprehensions, in the sense that if $\{\alpha\}: X \rightarrow A$ is a comprehension of α in $P(A)$, then $K(\{\alpha\}): KX \rightarrow KA$ is a comprehension of $k_A(\alpha)$.*

Proof. Since $P = (P)_x K^{op}$ and k has identity components, (K, k) preserves comprehensions. The rest follows immediately. \square

The results of this section, together with 4.5, produce an extension of the quotient completion of (Maietti and Rosolini 2012).

5.9 THEOREM. *There is a left bi-adjoint to the forgetful 2-functor from the full 2-category of **QED** on elementary doctrines with comprehensions, descent quotients and comprehensive diagonals into the 2-category **ED** of elementary doctrines.*

Proof. The left bi-adjoint sends an elementary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ to the elementary quotient completion $((P)_c)_x: \mathcal{X}_{((P)_c)_q}^{\text{op}} \rightarrow \mathbf{InfSL}$. \square

5.10 COROLLARY. *For $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ an elementary doctrine, the elementary quotient completion $\bar{P}: \mathcal{Q}_P^{\text{op}} \rightarrow \mathbf{InfSL}$ in (Maietti and Rosolini 2012) coincides with the doctrine $((P)_q)_x: \mathcal{X}_{(P)_q}^{\text{op}} \rightarrow \mathbf{InfSL}$.*

5.11 REMARK. Because of the logical setup in (Maietti and Rosolini 2012), only a particular case of 5.9 was proved, namely the left bi-adjoint was restricted to the full sub-2-category of **ED** of elementary doctrines with full comprehensions and comprehensive diagonals, see 5.3. On those doctrines $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$, the action of the left bi-adjoint was simply $((P)_q)_x: \mathcal{X}_{(P)_q}^{\text{op}} \rightarrow \mathbf{InfSL}$.

6 Comparing some free constructions

The elementary quotient completion resembles very closely that of exact completion. In fact, one has the following results.

6.1 THEOREM. *Given a cartesian category \mathcal{S} with weak pullbacks, let $\Psi: \mathcal{S}^{\text{op}} \rightarrow \mathbf{InfSL}$ be the elementary doctrine of weak subobjects. Then the doctrine $((\Psi)_q)_x: \mathcal{X}_{(\Psi)_q}^{\text{op}} \rightarrow \mathbf{InfSL}$, is equivalent to the doctrine $S: \mathcal{S}_{\text{ex}}^{\text{op}} \rightarrow \mathbf{InfSL}$ of subobjects on the exact completion \mathcal{S}_{ex} of \mathcal{S} .*

Proof. It follows from 4.3 and the characterization of the embedding of \mathcal{S} into \mathcal{S}_{ex} in (Carboni and Vitale 1998). \square

Though an elementary quotient completion with full comprehension is regular, see (Maietti and Rosolini 2012), the regular completion is an instance of a completion of a doctrine which is radically different from the elementary quotient completion in 5.9.

6.2 REMARK. For an elementary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$, a **weak comprehension of** α is an arrow $\{\alpha\}: X \rightarrow A$ in \mathcal{C} such that $\top_X \leq P_{\{\alpha\}}(\alpha)$ and, for every arrow $g: Y \rightarrow A$ such that $\top_Y \leq P_g(\alpha)$ there is a (not necessarily unique) $h: Y \rightarrow X$ such that $g = \{\alpha\} \circ h$, see (Maietti and Rosolini 2012).

For an elementary doctrine $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ with weak comprehensions, it is possible to add (strong) comprehensions to its extensional collapse as formal retracts of weak comprehensions: consider the category \mathcal{D}_P determined by the following data

objects of \mathcal{D}_P are triples (A, α, c) such that A is an object in \mathcal{C} , α is an object in $P(A)$, and $c: X \rightarrow A$ is a weak comprehension α

an arrow $[f]: (A, \alpha, c) \rightarrow (B, \beta, d)$ is an equivalence class of arrows $f: X \rightarrow Y$ in \mathcal{C} such that $P_{c \times c}(\delta_A) \leq P_{f \times f}(P_{d \times d}(\delta_B))$ with respect to the relation $f \sim f'$ determined by $P_{c \times c}(\delta_A) \leq P_{f \times f'}(P_{d \times d}(\delta_B))$

composition of $[f]: (A, \alpha, c) \rightarrow (B, \beta, d)$ and $[g]: (B, \beta, d) \rightarrow (C, \gamma, e)$ is $[g \circ f]$.

There is a full functor $K: \mathcal{C} \rightarrow \mathcal{D}_P$ defined on objects A as $K(A) := (A, \top_A, \text{id}_A)$ —it factors through \mathcal{X}_P . It preserves products and there is an extension $(P)_r: \mathcal{D}_P^{\text{op}} \rightarrow \mathbf{InfSL}$ of $P: \mathcal{C}^{\text{op}} \rightarrow \mathbf{InfSL}$ defined on objects as $(P)_r(A, \alpha, c) := \mathcal{D}es_{(P_c \times c)(\delta_A)}$. The doctrine $(P)_r: \mathcal{D}_P^{\text{op}} \rightarrow \mathbf{InfSL}$ is elementary with comprehensions and K preserves all existing comprehensions.

Given a cartesian category \mathcal{S} with weak pullbacks, let $\Psi: \mathcal{S}^{\text{op}} \rightarrow \mathbf{InfSL}$ be the elementary doctrine of weak subobjects. Then the doctrine $(\Psi)_r: \mathcal{D}_\Psi^{\text{op}} \rightarrow \mathbf{InfSL}$ is equivalent to the doctrine $S: \mathcal{S}_{\text{reg}}^{\text{op}} \rightarrow \mathbf{InfSL}$ of subobjects on the regular completion \mathcal{S}_{reg} of \mathcal{S} .

The proof is similar to that of 6.1 since, in the regular completion \mathcal{S}_{reg} of \mathcal{S} , every object is covered by a regular projective and a subobject of a regular projective.

Since the construction given in 6.1 factors through that in 6.2 via the exact completion of a regular category, see (Freyd and Scedrov 1991), and the exact completion of a weakly lex category may appear very similar to the category $\mathcal{X}_{((P)_q)_x}$, it is appropriate to mention an example of an elementary quotient completion which is not exact.

For that, consider the indexed poset on the monoid of partial recursive functions $F: \mathcal{N}^{\text{op}} \rightarrow \mathbf{InfSL}$ whose value on the single object of \mathcal{N} is the powerset of the natural numbers and, for any φ partial recursive function, $F_\varphi := \varphi^{-1}$, the inverse image of a subset along the partial map. It is clearly an elementary doctrine, and the doctrine $((F)_c)_x: \mathcal{X}_{(F)_c}^{\text{op}} \rightarrow \mathbf{InfSL}$ is equivalent to the subobject doctrine $S: \mathcal{PR}^{\text{op}} \rightarrow \mathbf{InfSL}$ on the category \mathcal{PR} of subsets of natural numbers and (restrictions of) partial recursive functions between them, see (Carboni 1995) for properties of that category, in particular its exact completion (as a weakly lex category) is the category \mathcal{D} of discrete objects of the effective topos.

Now, if one considers the elementary doctrine $((S)_q)_x: \mathcal{X}_{(S)_q}^{\text{op}} \rightarrow \mathbf{InfSL}$, the category $\mathcal{X}_{(S)_q}$ is equivalent to the category \mathcal{PER} of partial equivalence relations on the natural numbers, and the indexed poset $((S)_q)_x$ is equivalent to that of subobjects on that category. The category \mathcal{PER} is not exact because there are equivalence relations which are not equalizers. In fact, the exact completion $\mathcal{PER}_{\text{ex/reg}}$ of \mathcal{PER} as a regular category is the category \mathcal{D} of discrete objects.

Similar examples can be produced using topological categories such as those in the papers (Birkedal, Carboni, Rosolini, and Scott 1998; Carboni and Rosolini 2000). Other examples of elementary quotient completions that are not exact are given in the paper (Maietti and Rosolini 2012): one is applied to the doctrine of the Calculus of Constructions (Coquand 1990; Streicher 1992) and the other to the doctrine of the intensional level of the minimalist foundation in (Maietti 2009).

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